Answers to the Exam of Symmetry in Physics of April 2, 2013

Exercise 1

(a) Prove that the inverses of elements of a conjugacy class of a group G also form a conjugacy class of G.

> Define $K' = \{g^{-1} | g \in K\}$, where K is a conjugacy class of G. If K' is a class of G, then (1) all elements in K' are conjugated to each other and (2) no elements outside K' are conjugated to elements in K'. (1) follows from: $\forall h_1, h_2 \in K' : \exists h \in G : h_1^{-1} = h h_2^{-1} h^{-1}$, since $h_1^{-1}, h_2^{-1} \in K$. Therefore, $\forall h_1, h_2 \in K' : \exists h \in G : h_1 = h h_2 h^{-1}$. (2) follows from assuming $h_1 \in K', h_2 \notin K'$, but $\exists h \in G : h_1 = h h_2 h^{-1}$, and deriving a contradiction: it implies $h_1^{-1} = h h_2^{-1} h^{-1}$, hence $h_1^{-1} \in K$ is conjugated to h_2^{-1} which must also be in K therefore, and hence $h_2 \in K'$, contrary to the assumption.

(b) Show that the mapping $\phi : G \to G, g \mapsto g^{-1}$ is a 1-1 mapping that in general does not provide an isomorphism. Show for which type of groups it does provide an isomorphism.

> ϕ is a 1-1 mapping when $\forall g, h \in G : \phi(g) = \phi(h) \Rightarrow g = h$, which is the case since $g^{-1} = h^{-1} \Rightarrow g = h$. It is not an isomorphism, since it is not a homomorphism: $\phi(gh) = (gh)^{-1} = h^{-1}g^{-1} \neq g^{-1}h^{-1} = \phi(g)\phi(h)$. Only for an Abelian group it provides an isomorphism.

(c) Consider a conjugacy class K. Show by using Schur's lemma that $O = \sum_{g \in K} D(g)$ is proportional to the identity element, if D is an irrep.

> $\forall h \in G : D(h)O = \sum_{g \in K} D(hgh^{-1})D(h) = OD(h)$. The first step uses that D is a representation. The last step follows from the fact that $hgh^{-1} \in K$, hence $K' \equiv hKh^{-1} \subset K$, and since $g_i \neq g_j \Rightarrow hg_ih^{-1} \neq hg_jh^{-1}$ implies that K' has the same number of elements as K, one must have that $K' = K$. Therefore, the sum over all $g \in K$ can be replaced by a sum over hgh^{-1} which is again a sum over *all* elements of K.

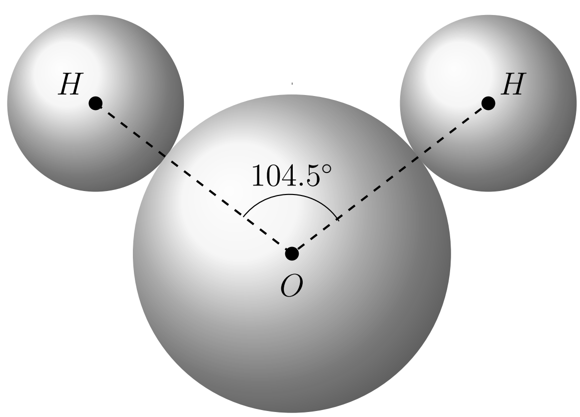
(d) Consider a regular n-sided polygon. Show that any rotation of the polygon is conjugated to its inverse rotation. Show this by using the defining relations of the group D_n and by geometrical arguments.

The symmetry group of a regular n-sided polygon is D_n. Its defining relation $(bc)^2 = e$ implies: $bcb = c^{-1}$, which together with $b^2 = e$ implies: $bcb^{-1} = c^{-1}$, which proves that $c \sim c^{-1}$. Similarly, $bc^m b = (bcb)^m = c^{-m}$. The geometrical argument uses the action of the symmetry group of the polygon embedded in R^3 on vectors in R^3. The rotation over an angle $2\pi m/n$, for any $m \in \{0, 1, \ldots, n-1\}$, is conjugated to the rotation over an angle $-2\pi m/n$, because the action on any 3-vector of both rotations can be related to each other by a rotation b over 180° (or in two dimensions a reflection in a line), which is also an element of the group.

(e) Consider $R \in O(3)$. Compute the determinant of the inverse of R.

$\det RR^T = 1 \Rightarrow (\det R)^2 = 1$ and $\det RR^{-1} = 1 \Rightarrow \det R^{-1} = 1/\det R$. Therefore, if $\det R = 1$, then $\det R^{-1} = 1$, and if $\det R = -1$, then $\det R^{-1} = -1$.

Exercise 2
Consider the water molecule H_2O:



(a) Determine the group G_W of all symmetry transformations that leave the water molecule invariant. (Hint: consider rotations *and* reflections in three dimensions.)

Consider the centers of the atoms to be in the x-y plane, with the x direction parallel to the H centers and the y axis orthogonal to it, running through the O center. Apart from the trivial operation e, the molecule is invariant under a rotation c around the y axis over 180°, a reflection b in the x-y plane, and a reflection (which is equal to bc) in the y-z plane. These operations form a group called $C_{2v} \cong D_2$: $gp\{b,c\}$ with $c^2 = b^2 = (bc)^2 = e$. The operations are clearly not conjugated to each other. They form an Abelian group.

(b) Construct the character table of this group G_W.

	(e)	(c)	(b)	(bc)
$D^{(1)}$	1	1	1	1
$D^{(2)}$	1	1	-1	-1
$D^{(3)}$	1	-1	1	-1
$D^{(4)}$	1	-1	-1	1

(c) Construct the three-dimensional vector representation D^V of G_W.

$$D^V(c) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D^V(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D^V(bc) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the determinants correctly correspond to rotations and reflections.

(d) Decompose D^V into irreps of G_W and use this to conclude whether the water molecule allows for an electric dipole moment or not.

$\chi^V = (3, -1, 1, 1)$, hence $D^V \sim D^{(1)} \oplus D^{(3)} \oplus D^{(4)}$, which follows either by direct inspection or by calculation of $\langle \chi^{(i)}, \chi^V \rangle$. The decomposition includes the trivial rep, which shows that there is an invariant vector/direction. The water molecule allows (and as you may recall from your chemistry class, it actually has) an electric dipole moment.

(e) Determine the Clebsch-Gordan series of the direct product representation $D^V \otimes D^V$ of G_W.

$\chi^{V \otimes V} = (9, 1, 1, 1)$, hence $D^{V \otimes V} \sim 3D^{(1)} \oplus 2D^{(2)} \oplus 2D^{(3)} \oplus 2D^{(4)}$, which follows either by direct inspection or by calculation of $\langle \chi^{(i)}, \chi^{V \otimes V} \rangle$.

(f) Explicitly determine the tensors T^{ij} $(i, j = 1, 2, 3)$ that are invariant under the transformations of G_W and check whether the answer is in agreement with the result obtained in part (e) of this exercise.

Find the 3×3 matrices T that commute with all $D^V(g)$ matrices by explicit computation:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

for any three independent numbers a, b, c. This means there are 3 independent invariant tensors, which correspond to the 3 trivial irreps present in the Clebsch-Gordan decomposition of $D^{V \otimes V}$ obtained in part (e).

Exercise 3

Consider the group of rotations in two dimensions $SO(2)$ and the unitary group $U(1)$.

(a) Write down the elements of $SO(2)$ in its defining representation.

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

These are all orthogonal 2×2 matrices with determinant 1.

(b) Write down the elements of $U(1)$ in its defining representation.

$$U(\phi) = (e^{i\phi})$$

These are all unitary 1×1 "matrices".

(c) Show that $SO(2) \cong U(1)$.

$R(\phi) \mapsto U(\phi)$ is a 1-1 mapping (recall that $e^{i\phi} = \cos\phi + i \sin\phi$) that is onto. Clearly they follow the same group multiplication: $R(\alpha)R(\beta) = R(\alpha + \beta)$, $U(\alpha)U(\beta) = U(\alpha + \beta)$.

(d) Write down all (complex) irreducible representations of $SO(2)$.

$D^{(m)}(\phi) = e^{im\phi}$, for all $m \in \mathsf{Z}$.

(e) Give an example of a physical system with an $SO(2)$ or $U(1)$ symmetry.

E.g. the rotation symmetry of a cylinder or the gauge symmetry of electromagnetic fields.

Next consider the extension of $SO(2)$ to include reflections: the group $O(2)$ of orthogonal 2×2 matrices.

(f) Write down the two-dimensional representation of $O(2)$ obtained by its action on the vector

$$\begin{pmatrix} x+iy \\ x-iy \end{pmatrix}$$

On this basis the elements of $SO(2)$ are:

$$R(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

Any other element of $O(2)$ can be written as $PR(\phi)$, for some P with $\det P = -1$, e.g. a reflection in the x axis, which on this basis is given by:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(g) Show whether this two-dimensional rep of $O(2)$ is an irrep or not.

One way is to note that the matrices of $O(2)$ cannot simultaneously be brought to the form:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Another way is to note that $PR(\phi)P = R(-\phi)$, hence the only matrix that commutes with all $O(2)$ matrices is $R(\phi = 0)$, which is the identity matrix. This also implies that the 2-D rep is an irrep.